Problem 1. Mechanics on curved space

Here you will explore classical mechanics on curved spaces, in arbitrary coordinate systems. We'll introduce crucial geometric ideas (we won't actually need them in this problem, but I want to introduce them to help you recognize connections between mechanics and geometry).

Christoffel symbols are fundamental objects that tell you how basis vectors change from one point in space to another. Letting $\vec{\epsilon}_\mu$ be the μ'th covariant basis vector, they are

$$\frac{\partial \vec{\epsilon}_k}{\partial q^j} = \sum_\mu \Gamma^\mu_{jk} \vec{\epsilon}_\mu. \tag{1}$$

In other words, the Γ^μ_{jk} tell us how the k'th basis vector change when you change q_j. (Γ looks like a tensor, but is not.) This immediately gives

$$\Gamma^\mu_{jk} = \vec{\epsilon}^\mu \cdot \frac{\partial \vec{\epsilon}_k}{\partial q^j}. \tag{2}$$

Since g^{ij} encodes the dot product of basis vectors, it is unsurprising the rhs is related to a derivative of g. After a little algebra, (go through the derivation in Sec 4.3 of A&W):

$$\Gamma^\mu_{ij} = \frac{1}{2} \sum_k g^{\mu k} \left[\frac{\partial g_{ik}}{\partial q_j} + \frac{\partial g_{jk}}{\partial q_i} - \frac{\partial g_{ij}}{\partial q_k} \right]. \tag{3}$$

These are important since they not only tell us how the basis vectors vary, they also provide a way to take the geometric derivative of a vector. For example, to determine how a vector field changes in space, you cannot simply partially differentiate its coordinates. As a vivid example, even a geometrically constant vector can have components that vary in space (consider e.g. polar coordinates). So you need to extricate the geometric variation of the vector from that coming from the mere coordinate-dependence of its components. The Christoffel symbols let us differentiate vectors accounting for this (*covariant derivatives*, which we may encounter later).

In this problem, you consider some classical mechanics in arbitrary coordinates and/or curved space, and see the Γ provide the relevant information. Consider a particle in a potential $V(\vec{r})$ with kinetic energy $T = \sum_j \dot{x}_j^2$, where x_j are Cartesian coordinates.

(a) For an arbitrary coordinate system $\{q_j\}$, where each q_j is an arbitrary (time-independent, i.e. *natural coordinates*) function of the $\{x_j\}$, what is T? Write this in terms of the metric g_{ij}.

(b) What are the equations of motion for this particle? Write in the form

$$\ddot{q}_j = (\text{stuff depending on metric, its derivatives with respect to } q_j\text{'s, and } V) = 0. \tag{4}$$

(*Hint: They are given by the Euler-Lagrange equations for the Lagrangian* $L = T - V$.)

(c) Rewrite the equations of motion you obtained in terms of the Christoffel symbols. The metric should not appear.

(d) Determine the Christoffel symbols and write out these equations of motion for a 2D particle in polar coordinates.

(e) When the potential V vanishes, the equations of motion are the equations of *geodesics*, curves that trace the shortest path between two points. In flat space, a line is the shortest distance between two points. Let's see this works in polar coordinates (a perverse coordinate choice for the problem). What is the equation for a geodesic (i.e. a line) in polar coordinates? Show that the solutions to the equations of motion give a geodesic in this case.

(f) Now solve these equations for a particle on the surface of a unit sphere ($r = 1$), with initial velocity of magnitude $|v| = 1$ along the equator.

A moral: modifications arising from arbitrary coordinates/curved space, automatically captured by the Lagrangian formalism, are related to quantities capturing the geometry of the surface.

Problem 2. Two first-order 2D PDEs

(a) Consider

$$\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} - (x - y)\phi = 0. \tag{5}$$

What are its characteristics? Show that $\phi = e^{-xy} f(x+y)$ is a solution for f an arbitrary function.

(b) Consider the 2D PDE (note: non-constant coefficients!):

$$y\frac{\partial \phi}{\partial x} + x\frac{\partial \phi}{\partial y} = 0. \tag{6}$$

What are its characteristics? Given initial data the function $\phi(x, 0)$, what is $\phi(x, y)$. You may consider the region $y > x > 0$ if helpful. *Hint: After solving the ODE along characteristics, you may find it helpful to consider the quantity* $x^2 - y^2$ *to simplify.*

Problem 3. A second order PDE

(a) Show that for a constant-coefficient second order PDE ($D\phi = 0$, with D the short hand for the relevant differnetial operator) where D can be factored into two distinct factors $D = P_1 P_2$ with

$$P_i = \alpha_i \frac{\partial}{\partial x} + \beta_i \frac{\partial}{\partial y} + \gamma_i, \tag{7}$$

then $D\phi = 0$ has the general solution $\phi = \phi_1 + \phi_2$, where $P_1\phi_1 = 0$ and $P_2\phi_2 = 0$.

(b) Show that the general solution to

$$\frac{\partial^2 \phi}{\partial x \partial y} + 2\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial x} - 2\frac{\partial phi}{\partial y} = 0 \tag{8}$$

is

$$\phi(x, y) = f(2x - y) + e^y g(x). \tag{9}$$